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INTRODUCTION TO DYNAMICS OF STONE ARCHES

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SUMMARY

The dynamic analysis of stone arches, made up of rigid voussoir laid dray, can be performed in two phases. First of all the value of the horizontal acceleration necessary to turn the structure into a mechanism and the corresponding mechanism must be determined. Then the dynamic behaviour of such a mechanism under a given acceleration time history can be studied. The first step is a static matter. The second one requires the solution of the non-linear equation of motion of the one-degree-of-freedom system in which the arch is turned. In this paper an iteration procedure is proposed to find out the mechanism. Then the structural behaviour of the mechanism is analysed. Both free and forced vibrations are investigated and the study is limited to the first-half cycle of vibration. Damping is not considered and sliding between the blocks at the hinge sections is not allowed. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: stone arch; non-linear dynamic analysis; seismic analysis of stone arches; dynamics of rigid bodies

INTRODUCTION

The analysis of stone arches is usually performed by referring to the Heyman model, made up of rigid voussoirs laid dray. The limit domain in the plane M–N (bending moment — axial force) of the rectangular cross-section between two voussoirs is given by the two straight lines starting from the origin:[1,2]

$$M = \pm N s/2 \tag{1}$$

s being the actual thickness of the arch. Friction between voussoirs is supposed to be high enough to avoid sliding failure.

Failure of the arch occurs when sufficient hinges form to turn the structure into a mechanism. In other words if a line of thrust lying wholly within the masonry which represents an equilibrium state for the structure under the action of the external loads and which allows the formation of sufficient hinges to transform the structure into a mechanism can be found, then the arch is on the point of collapse.

The application of the ideas of the plastic theory to stone arches was already discussed and the structural behaviour under vertical loads was investigated.[3,4]

The study of the dynamics of rigid bodies started from a famous paper by Housner[5] on the dynamic behaviour of a rocking block. His concepts were used and developed by different authors to describe the rocking response of rigid bodies to earthquake excitation.[6,7] A large bibliography on the subject was given by Augusti and Sinopoli.[8] Relevant contributions to the understanding of the dynamic response of rigid blocks were given by Tso and Wong[9] and Hogan.[10]

Nowadays, the non-linear analysis of stone arches can be performed by means of computer codes, by using complex finite element models. However, in order to investigate the basic aspects of the dynamic behaviour

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of stone arches, it is useful to refer, for a preliminary analysis, to the model proposed by Heyman. This is rigorously suitable only for stone voussoir arches, but it must be pointed out that any good masonry arch behaves, after cracking, like a voussoir arch laid dray as shown by the experimental analysis carried out on shake table.[11]

The dynamic behaviour of a stone voussoir arch is very complex, because of the highly non-linearity. Therefore, it is a very interesting problem from a theoretical point of view, but it is studied also for technical reasons. Among these are the evaluation of the dynamic response of monumental structures and therefore an estimate of the ground shaking intensity of past earthquakes.[9]

In this paper the dynamic behaviour of a stone arch under base motion is analysed in two steps. In the first one, relative to the onset of motion, the static analysis is carried out to find the collapse mechanism and the corresponding horizontal acceleration factor. In the second step the structural behaviour of the four link mechanism is analysed. Both free and forced vibrations are analysed and the study is limited to the first-half cycle of vibration. Damping is not considered and sliding between the blocks at the hinge sections is not allowed.

ONSET OF MOTION

Consider the voussoir circular arch in Figure 1, of angle of embrace β and thickness s. Suppose that one thrust line in equilibrium with the dead loads exists, which lies wholly within the masonry. The base of the arch is subject to an horizontal acceleration $-\ddot{x}_g = -\lambda g$, g being the gravity acceleration. In this condition the arch is subjected to the vertical loads and to the horizontal inertial forces due to the acceleration. According to the safe theorem the structure is safe if a line of thrust in equilibrium with the external loads and lying wholly within the masonry can be found.

The uniqueness theorem can be stated as follows: if the absolute value of the acceleration, and so the forces of inertia, is increased to the collapse value, while the vertical load does not change, the acceleration value on the point of collapse is unique. The line of thrust changes and at least four hinges form. At the point of collapse the thrust line must pass through the hinge points and if the hinge is at an internal cross-section it must be tangential to the arch profile.

The collapse mechanism and the collapse factor can be found by using an iteration procedure,[4] that can be started giving a first mechanism and the corresponding diagrams of the velocity components (Figure 2(a)). The equation of equilibrium can be written as

$$P_v + \lambda \cdot P_h = 0 \tag{2}$$

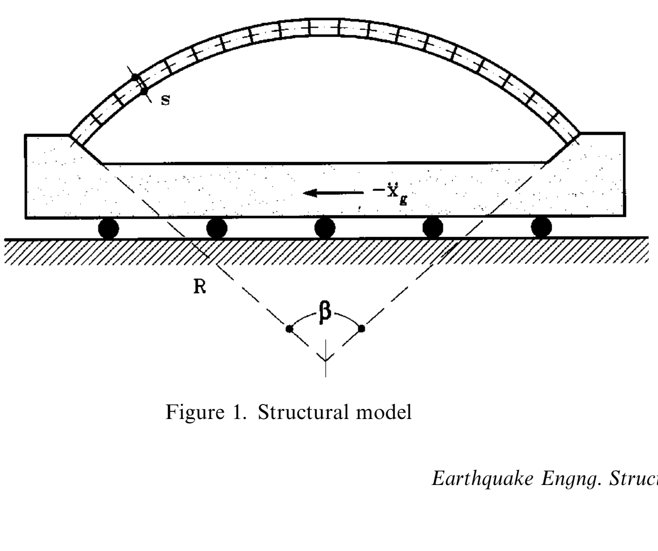

Figure 1. Structural model

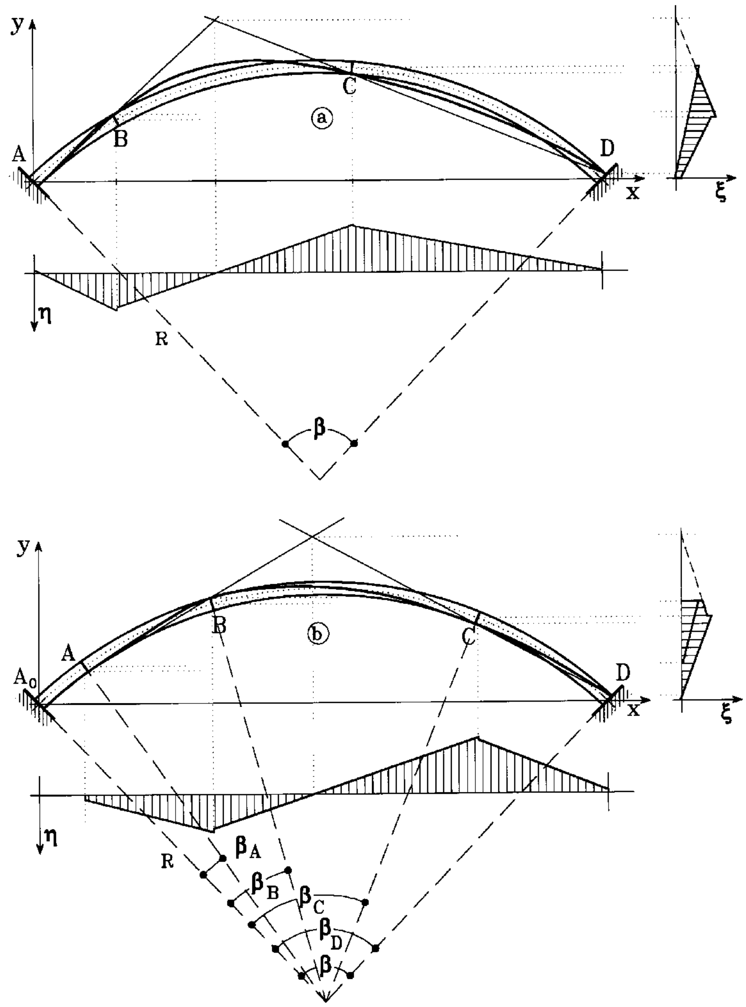

Figure 2. Thrust line in the generic iteration (a) and actual line of thrust on the point of collapse (b)

where P_v and P_h are the virtual powers due to vertical and horizontal loads, respectively,

$$P_v = \int_0^L m(x) \cdot g \cdot \eta \, dx, \qquad P_h = \int_0^L m(x) \cdot g \cdot \xi \, dx \tag{3}$$

In equation (3) m is the mass per unit span length, η and ξ are the vertical and the horizontal components of the velocity, respectively (Figure 2). As one can see the vertical loads have a stabilising effect. In fact, because in equation (2) there are no terms with the internal forces acting at the hinges, which are still unknown, the equilibrium on the point of collapse is guaranteed by P_v only. It must be $P_v < 0$. From equation (2) the *kinematically admissible* acceleration factor can be deduced

$$\lambda = -\frac{\int_0^L m \cdot g \cdot \eta \, dx}{\int_0^L m \cdot g \cdot \zeta \, dx} \tag{4}$$

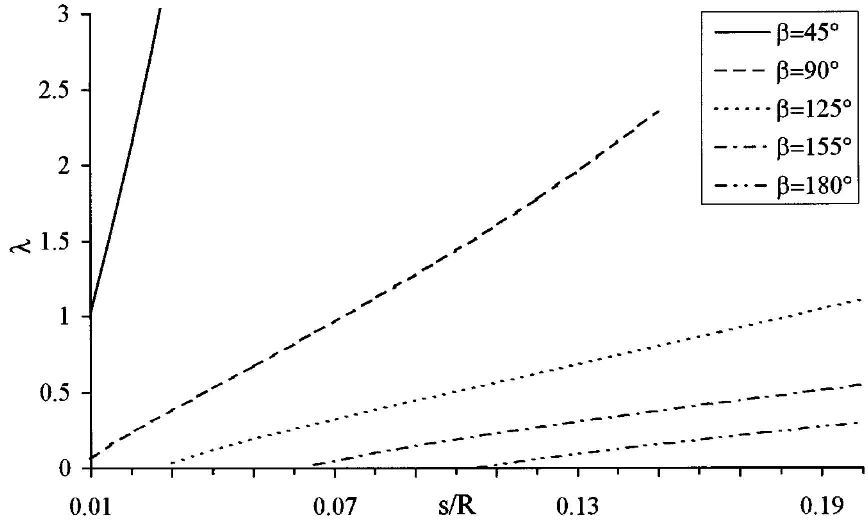

Figure 3. Onset of motion: λ versus s/R for different values of β

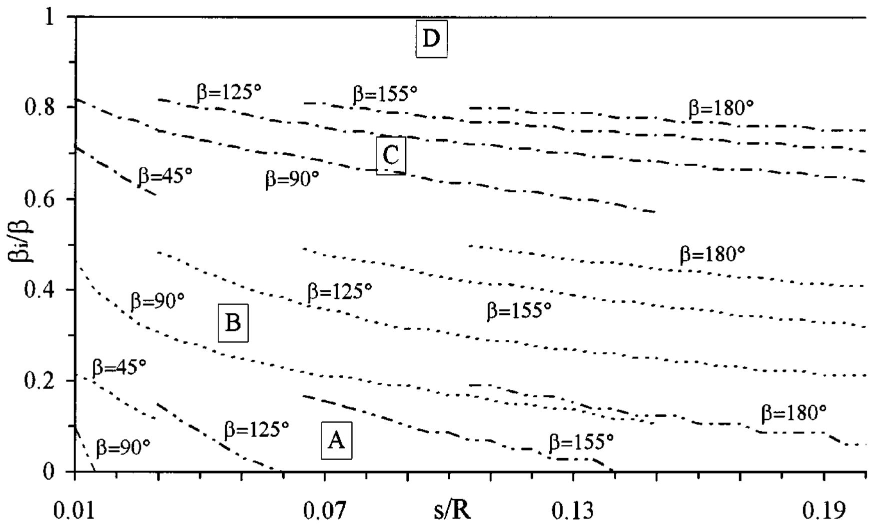

Figure 4. Onset of motion: hinge locations β_i/β versus s/R for different values of β

From λ the load acting on the structure can be found out and then the reactions and the funicular polygon passing through the assumed hinges can be determined. The value given by equation (4) is the collapse acceleration factor if the funicular polygon passing through the hypothesised hinges is elsewhere within the masonry. If it is not, in the next step of the iteration procedure we must move the hinges to the sections in which the distances between the funicular polygon and the arch profile are maximum (Figure 2(b)).

In Figure 3 the acceleration factor λ necessary to turn the structure into a mechanism versus the ratio s/R is plotted, for different values of β. As one can see λ increases with s/R with linear laws, for each β. Besides, λ gets very high when β decreases. This occurrence guarantees that during the oscillation, which follows the onset of motion, each bar behaves as a rigid body and no sub-mechanism forms. In fact, each bar is an arch with a small β and, therefore, a very high value of λ is necessary to turn it into a mechanism, even though the

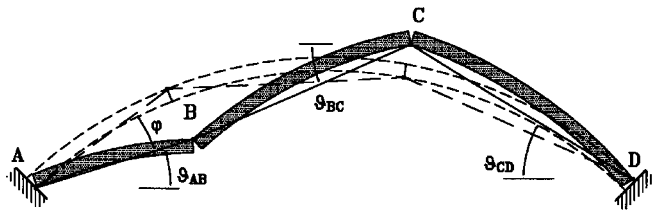

Figure 5. Collapse mechanism

springing are at different heights. In Figure 4 the hinge locations β_i/β (i = A, B, C, D), for different values of β, are shown. In circular arches hinge D forms always at the right springing, while hinge A may not form at the left springing.

DYNAMIC BEHAVIOUR

If the acceleration value is higher than the limit value before found, then the structure is turned into a mechanism (Figure 5). Its dynamic behaviour is influenced very much by the damping mainly due to the impacts at the hinge sections, when the structure passes through its natural configuration. In the following investigation, the attention has been focused on the first-half cycle of vibration, up to the first return to the natural configuration, and the damping has not been considered. The equation of motion of the SDOF system,[12] has been numerically solved assuming the rotation φ of bar AB as Lagrangian co-ordinate (Figure 5).

A circular arch with $\beta = 125^\circ$ and $s/R = 0{\cdot}06$ has been considered. The structure has been supposed to be formed by a high number of voussoir. Hinges form at springing (A and D) and at sections at -15° (B) and $+34^\circ$ (C) from the crown, when the horizontal acceleration is equal to $0{\cdot}263g$.

Free vibrations

Let us start by analysing the free vibrations of a voussoir arch with initial conditions $\varphi(0) = \varphi_0$ and $\dot{\varphi}(0) = 0$. Obviously, if φ_0 is greater than a certain value φ_v then the structure will collapse. If $\varphi_0 < \varphi_v$ it will return to the natural configuration, the limit value φ_v corresponding to a maximum value for the potential energy. At $\varphi = \varphi_v$ the equilibrium is unstable: if $\varphi_0 = \varphi_v$ the arch will remain in this configuration indefinitely, only in absence of disturbances. In Figure 6 the diagrams of φ_v versus s/R for different values of β are plotted. As one can see, φ_v increases with s/R and gets higher when β decreases. For the analysed arch ($\beta = 125^\circ$ and $s/R = 0{\cdot}06$) it is $\varphi_v = 0{\cdot}0385$ rad. If $\varphi_0 < \varphi_v$ it is meaningful to study the dynamic behaviour of the arch for $t > 0$. In Figure 7 the curves $\varphi(t)$ for different initial rotation angle φ_0 are shown. Obviously, the time the structure takes to return to its natural configuration increases non-linearly with φ_0.

Rectangular pulse base acceleration

Consider the arch subject to a rectangular pulse base acceleration of amplitude a and time duration t_d. If the acceleration value a is large enough to turn the structure into a mechanism, then the rotation angle φ increases for $t < t_d$, up to a value $\varphi(t_d)$. If $\varphi(t_d) > \varphi_v$ the arch will certainly collapse. If $\varphi(t_d) < \varphi_v$ the structure may or may not collapse: it depends on the rotation angle and velocity at time t_d.

In Figure 8, the diagram of the acceleration value a_c for which the arch will collapse versus t_d is plotted. If $a < 0{\cdot}263g$, the horizontal acceleration is not sufficient to turn the structure into a mechanism and the arch will stay in its natural configuration. As we can see a_c decreases rapidly when t_d gets higher. The curve splits the plane into two portions. The couples of values (a, t_d) for which the arch will collapse are located above the curve. It is interesting to point out that if $t_d \to 0$ then the value of the acceleration necessary to bring the structure to collapse becomes very large.

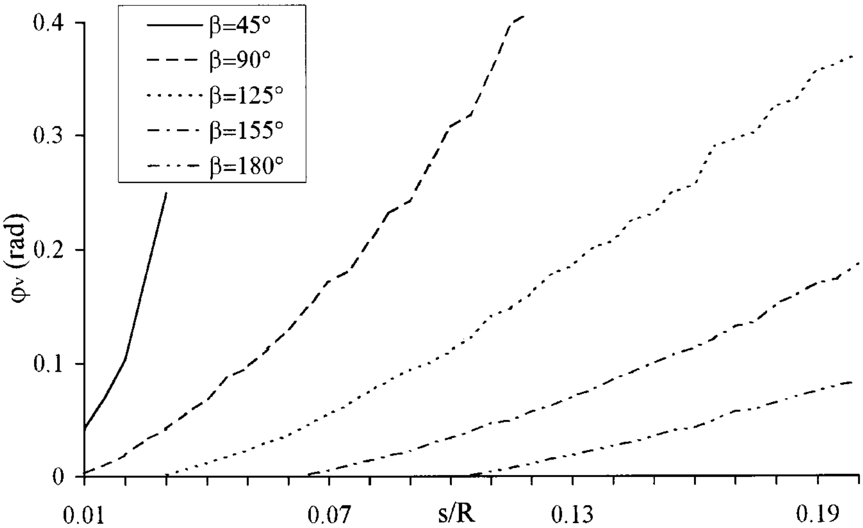

Figure 6. Free vibrations: φ_v versus s/R for different values of β

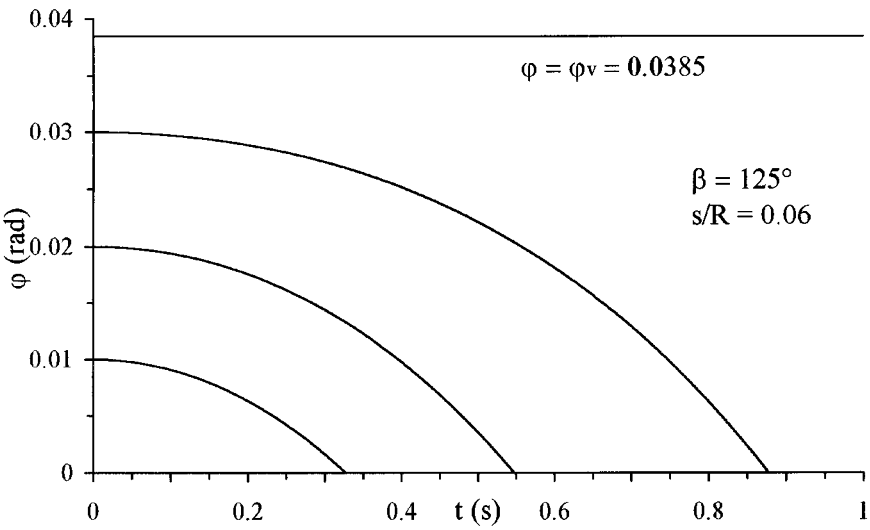

Figure 7. Free vibrations: $\varphi(t)$ for different values of φ_0

If a return pulse is present, then the arch can temporarily achieve rotation $\varphi > \varphi_v$, if the return motion restores the structure to a range below φ_v.

Sinusoidal base acceleration

Consider the stone arch subject to the sinusoidal base acceleration $a \sin(2\pi/T_a)t$, with $0 \leqslant t \leqslant t_d$, of frequency $1/T_a$ and amplitude a. In Figure 9 the amplitude a_c, for which the structure collapses, is plotted

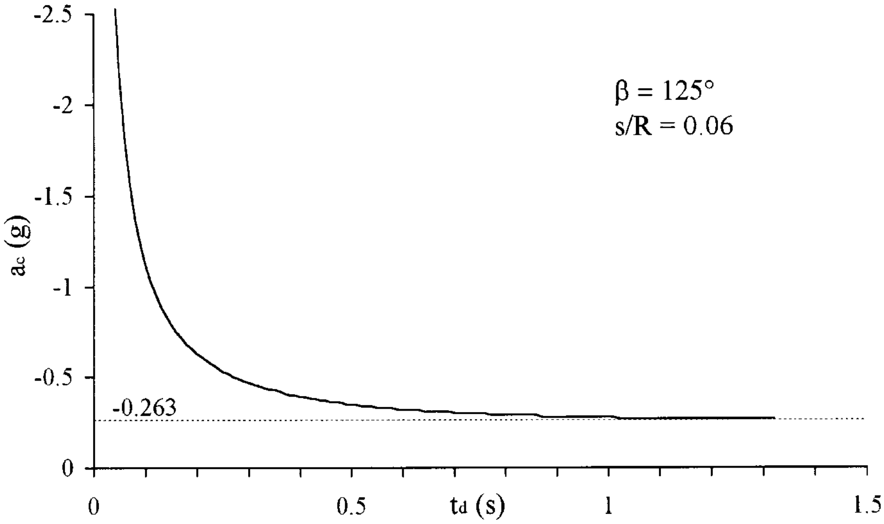

Figure 8. Rectangular pulse base acceleration: λ versus t_{d}

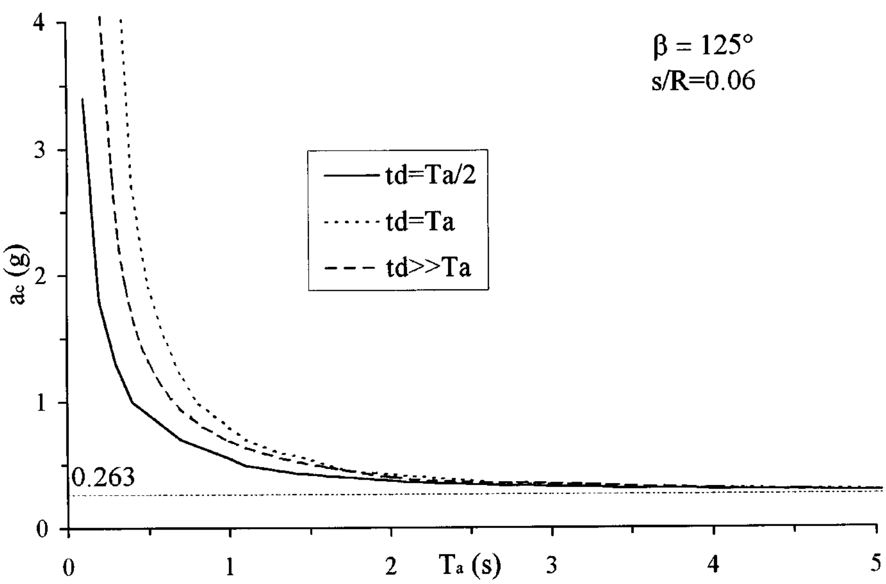

Figure 9. Sinusoidal base acceleration; λ versus T_a for $t_{\mathrm{d}} = T_a/2$, $t_{\mathrm{d}} = T_a$ and $t_{\mathrm{d}} \gg T_a$

versus T_a, for the two cases of a half-cycle sine pulse acceleration ($t_{\mathrm{d}} = T_a/2$) and a cycle sine acceleration ($t_{\mathrm{d}} = T_a$). Obviously, the second curve shows values larger than those of the first one. The minimum value $0{\cdot}263g$ of the ground acceleration, necessary to turn the structure into a mechanism, coincides with the collapse value a_{c} only for high values of T_a, both for $t_{\mathrm{d}} = T_a/2$ and $t_{\mathrm{d}} = T_a$. The collapse value a_{c} increases rapidly when T_a decreases. This occurrence seems obvious for the considered simple input. Actually, the consideration that the structure is more vulnerable to low-frequency input is valid in general. In the following analysis, several sinusoidal acceleration time-histories, characterized by different values of the frequency, have been considered. For each of them the time-history $\varphi(t)$ has been plotted.

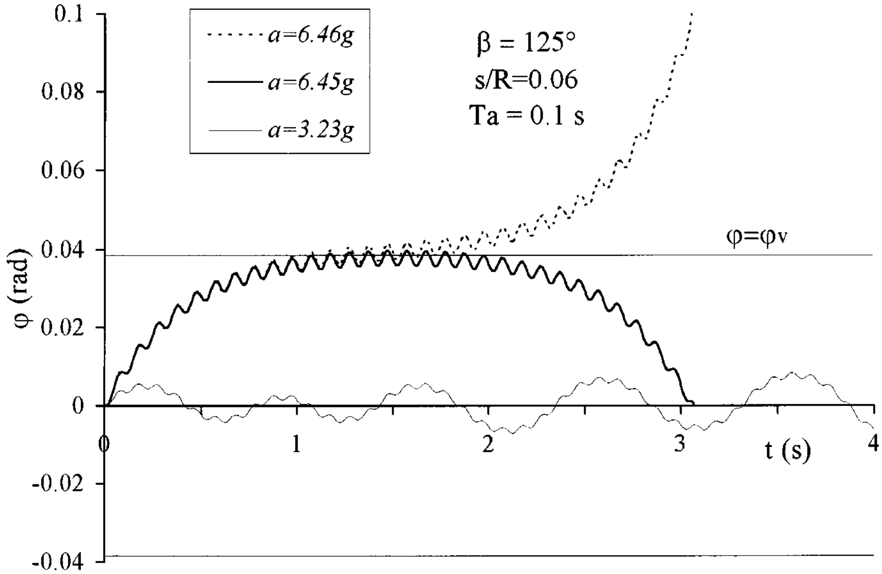

Figure 10. Sinusoidal base acceleration: $\varphi(t)$ for $T_a = 0{\cdot}1$ sec (different values of a)

In Figure 10 the diagram $\varphi(t)$ relative to a sinusoidal base acceleration with $T_a = 0{\cdot}1$ sec is shown. In particular, the two cases of $a = 6{\cdot}45g$ and $a = 6{\cdot}46g$ are considered, the second being the minimum value a_c for which the structure does not return to its natural configuration. For lower value of a ($a = a_c/2$ in the figure) the system would continue its oscillations around the natural configuration. The corresponding time history is plotted in Figure 10 under the hypothesis of arch turning into mechanism A–B–C–D and into mechanism A′–B′–C′–D′ (in which A′ is placed symmetrically to A around the crown and so on) alternatively. The dissipation of energy during the impacts is not considered. Rotation φ is relative to bar A′B′ when $\varphi < 0$. It is important to point out that, after some oscillations, the arch may collapse even for acceleration value lower than a_c, the time duration of the base input playing a fundamental role.

The structural behaviour is clear. A higher-frequency component, due to the input acceleration ($1/T_a = 10$ Hz), is apparent in the diagram. If $a < a_c$ the structure will return to its natural configuration in a time that depends on the geometrical characteristics and on the acceleration amplitude. In particular it gets higher with a.

In Figures 11 and 12, the time-history responses relative to sinusoidal base accelerations with $T_a = 1{\cdot}0$ and 5·0 sec, respectively, are plotted. As already said, the collapse value a_c decreases when T_a increases and, for $T_a > 5{\cdot}0$ sec, it is practically coincident with the minimum one necessary to turn the structure into a mechanism. From Figures 10 and 11 it is also apparent that the arch can cross the configuration in which $\varphi = \varphi_v$ and then return to its natural configuration.

The diagram of the collapse value a_c versus T_a, for a sinusoidal base acceleration input with $t_d \gg T_a$, is also plotted in Figure 9. The effect of the duration of a sinusoidal base acceleration is evident. In fact, for a given T_a, the collapse value a_c is minimum for $t_d = T_a/2$ and maximum for $t_d = T_a$. If $t_d > T_a$ the collapse value is certainly lower than that relative to $t_d = T_a$. This behaviour is apparent in the time history of Figure 11, in which for $a = 0{\cdot}69g$ the arch collapses if $t_d = T_a/2$ or $t_d = 2T_a$, but does not fail if $t_d = T_a$.

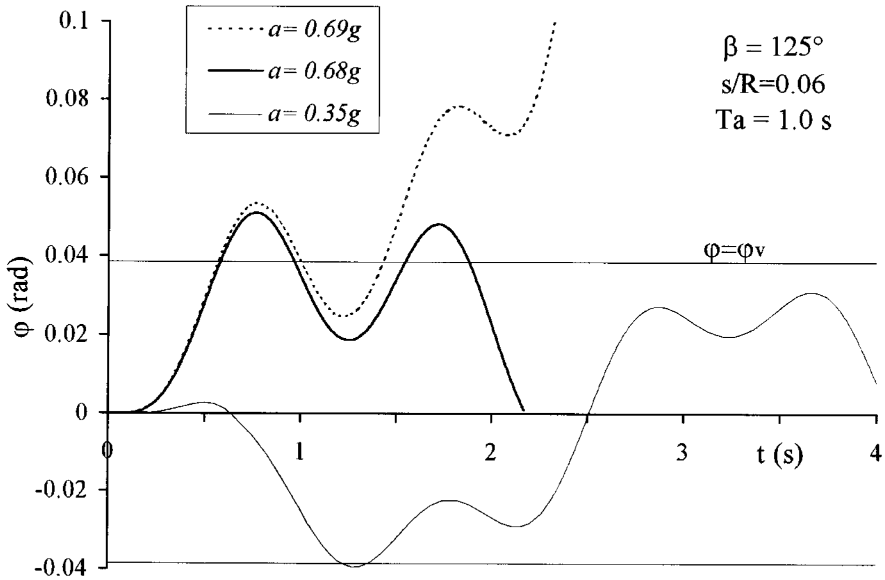

Figure 11. Sinusoidal base acceleration: $\varphi(t)$ for $T_a = 1$ sec (different values of a)

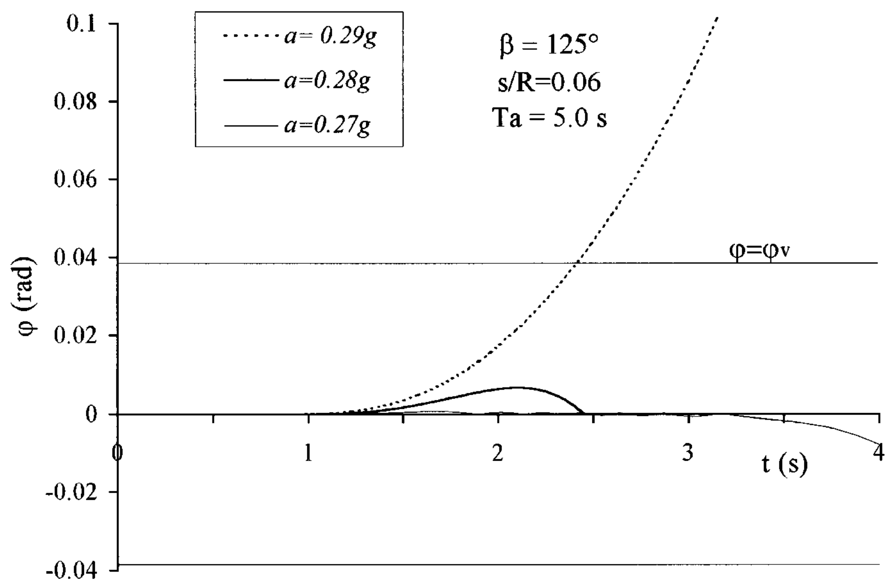

Figure 12. Sinusoidal base acceleration: $\varphi(t)$ for $T_a = 5$ sec (different values of a)

CONCLUSIONS

The seismic check of a stone voussoir arch can be performed by comparing the expected Peak Ground Acceleration (PGA) with the acceleration value necessary to turn the structure into a mechanism. If PGA $< \lambda g$ then a line of thrust in equilibrium with the external loads and lying wholly within the masonry

exists and the arch will remain in its natural configuration. From a technical point of view this is enough to state the safety of a stone voussoir arch subject to seismic actions. A new stone arch must satisfy this condition.

If PGA $\geqslant \lambda g$ then the structure will be turned into a mechanism. The numerical investigation carried out in the paper showed that the dynamic response of a stone voussoir arch depends very much on the input characteristics, such as frequency content and amplitude. It also depends on the initial conditions.

Inputs with low period T_a requires high values of the amplitude to bring the structure to the collapse. If T_a gets higher the value of the amplitude of a sinusoidal input acceleration for which the arch collapses approaches to the minimum necessary to turn it into a mechanism.

It is important to underline that the results here shown give only a first approximation of the dynamic response of a stone arch under base motion. In order to improve the analysis and to study the seismic response of such a structure, the effects of the impacts between adjacent blocks must be analysed. In fact, after each impact the response changes due to the energy dissipation. The collapse mechanism may change as well. Finally, the structural behaviour is also influenced by the actual strength of the material, which is limited and not infinite, as supposed in the Heyman model.

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